Achieving Optimal Revenue with Enhanced Competition

Linda Cai

General Exam at Princeton University

Overview of My Research

Auction Design

- Prove that simple auction can achieve 99% of optimal revenue with constant enhanced competition [EC21]

  Joint work with Raghuvansh Saxena

- Implementation in advised strategies: a new solution concept for self interested behavior when being truthful is NP-hard [ITCS20]

  Joint work with Clayton Thomas and Matt Weinberg

- Repeated auction design for buyers using no regret learning algorithms

  Joint work with Matt Weinberg, Evan Wildenhain and Shirley Zhang

Stable Matching

- A simple proof for short-side advantage in random matching markets [SOSA21]

  Joint work with Clayton Thomas and Matt Weinberg

Overview of My Research

Auction Design

- **99% Revenue with Constant Enhanced Competition [EC21]**
  Joint work with Raghuvansh Saxena
- Implementation in advised strategies: a new solution concept for self interested behavior when being truthful is NP-hard [ITCS20]
  Joint work with Clayton Thomas and Matt Weinberg
- Repeated auction design for buyers using no regret learning algorithms
  Joint work with Matt Weinberg, Evan Wildenhain and Shirley Zhang

Stable Matching

- A simple proof for short-side advantage in random matching markets [SOSA21]
  Joint work with Clayton Thomas and Matt Weinberg

Outline

Goal: design truthful auction that gets as much revenue as possible

Outline

Goal: design truthful auction that gets as much revenue as possible

- Maximizing revenue is easy when there is one item

Outline

Goal: design truthful auction that gets as much revenue as possible

- Maximizing revenue is easy when there is one item
- Maximizing revenue is hard when there are multiple items

Outline

Goal: design truthful auction that gets as much revenue as possible

- Maximizing revenue is easy when there is one item
- Maximizing revenue is hard when there are multiple items
- Enhanced competition: can a simple auction achieve the optimal revenue by recruiting more bidders

Outline

Goal: design truthful auction that gets as much revenue as possible

- Maximizing revenue is easy when there is one item
- Maximizing revenue is hard when there are multiple items
- Enhanced competition: can a simple auction achieve the optimal revenue by recruiting more bidders
- Our result: simple auction can achieve 99% of the optimal revenue with constant enhanced competition

Revenue Maximizing Auction

Combinatorial auction: n bidders, m items.

- Each bidder i has valuation function $v_i : 2^m \to R^+$.
- Bidders participate in some (possibly interactive) protocol.
- Auctioneer awards the set of items S_i to bidder i, charges price p_i.

Revenue Maximizing Auction

Combinatorial auction: n bidders, m items.

- Each bidder i has valuation function $v_i : 2^m \to R^+$.
- Bidders participate in some (possibly interactive) protocol.
- Auctioneer awards the set of items S_i to bidder i, charges price p_i.

Bidder Goal: Maximizes (expected) utility $= v_i(S_i) - p_i$

Revenue Maximizing Auction

Combinatorial auction: n bidders, m items.

- Each bidder i has valuation function $v_i : 2^m \to R^+$.
- Bidders participate in some (possibly interactive) protocol.
- Auctioneer awards the set of items S_i to bidder i, charges price p_i.

Bidder Goal: Maximizes (expected) utility $= v_i(S_i) - p_i$

Auctioneer Goal: Maximizes (expected) revenue $= \sum_i p_i$.

Revenue Maximizing Auction

Combinatorial auction: n bidders, m items.

- Each bidder i has valuation function $v_i : 2^m \to R^+$.
- Bidders participate in some (possibly interactive) protocol.
- Auctioneer awards the set of items S_i to bidder i, charges price p_i.

Bidder Goal: Maximizes (expected) utility $= v_i(S_i) - p_i$

Auctioneer Goal: Maximizes (expected) revenue $= \sum_i p_i$.

Bidder goal different from auctioneer goal, how can the auctioneer predict bidder behavior?

Revenue Maximizing Auction

Combinatorial auction: n bidders, m items.

- Each bidder i has valuation function $v_i : 2^m \to R^+$.
- Bidders participate in some (possibly interactive) protocol.
- Auctioneer awards the set of items S_i to bidder i, charges price p_i.

Bidder Goal: Maximizes (expected) utility $= v_i(S_i) - p_i$

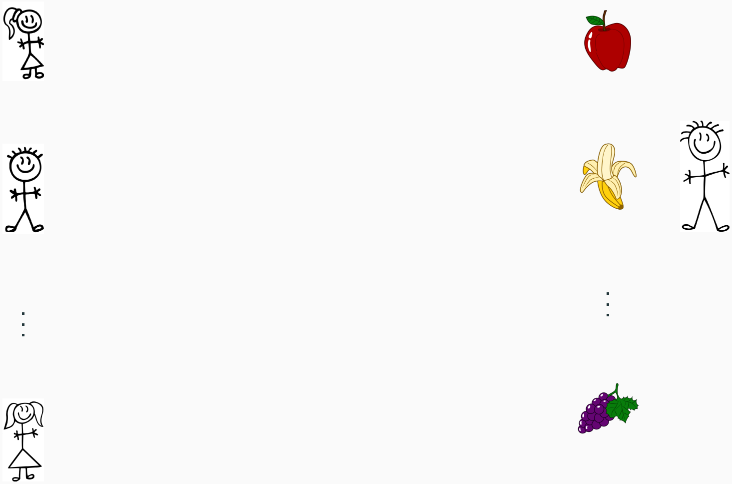
Auctioneer Goal: Maximizes (expected) revenue $= \sum_i p_i$.

Bidder goal different from auctioneer goal, how can the auctioneer predict bidder behavior?

Truthful Auction (Informal)
An auction is *truthful* if it is in the bidder's best interest to behave truthfully (e.g. bidding their own value)

Auctioneer Constraint: Use truthful auctions

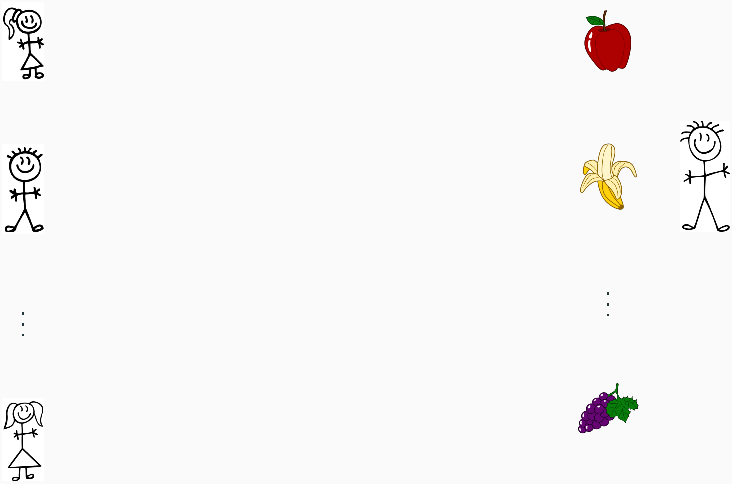
Revenue Maximizing Auction: Our Setting



n i.i.d additive bidders, m items

Revenue Maximizing Auction: Our Setting

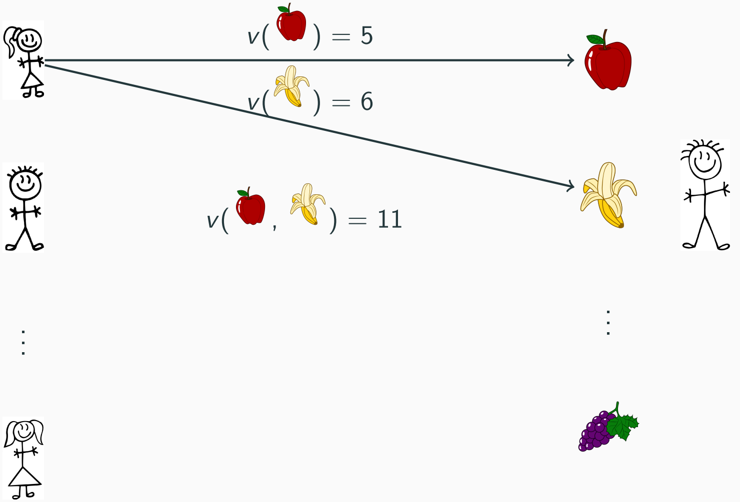
$v_i \sim \mathcal{D} = \times_j \mathcal{D}_j$



n i.i.d additive bidders, m items

Revenue Maximizing Auction: Our Setting

$v_i \sim \mathcal{D} = \times_j \mathcal{D}_j$

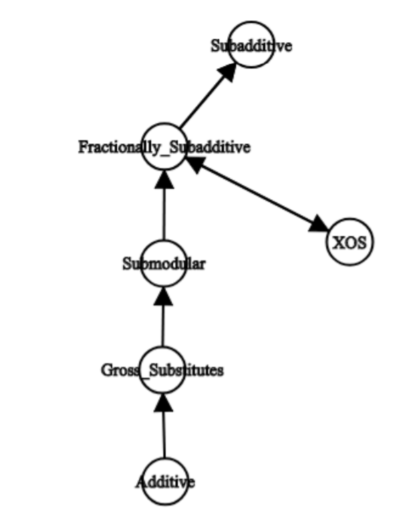


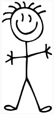
n i.i.d additive bidders, m items

Revenue Maximizing Auction: Our Setting

$v_i \sim \mathcal{D} = \times_j \mathcal{D}_j$

$\vdots$



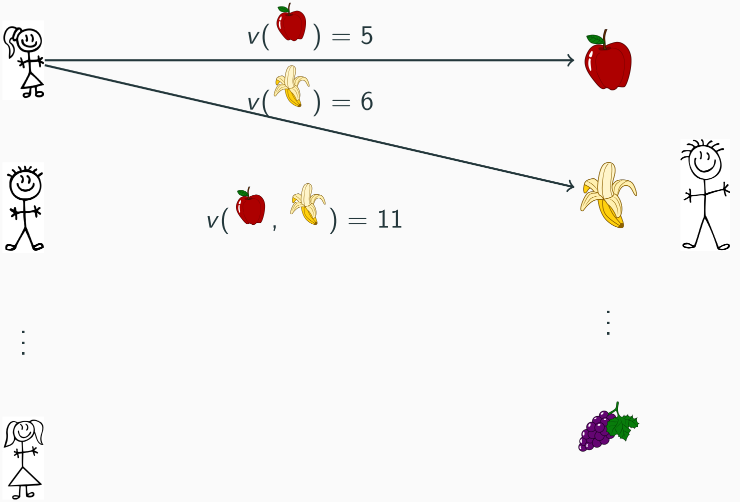


$\vdots$

n i.i.d additive bidders, m items

Revenue Maximizing Auction: Our Setting

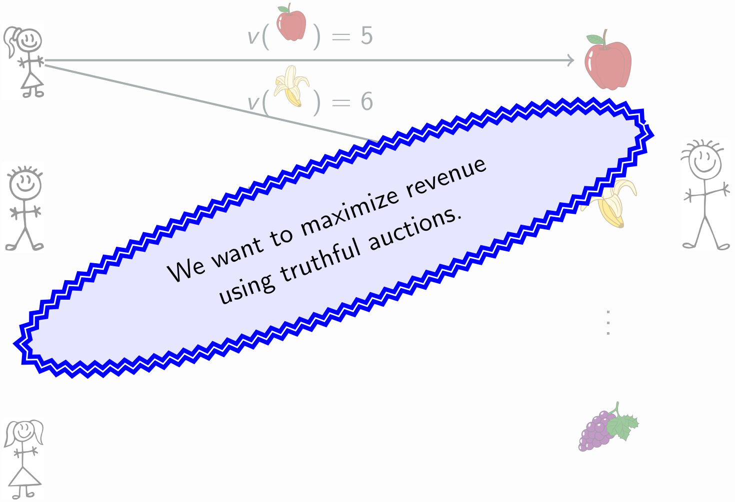
$v_i \sim \mathcal{D} = \times_j \mathcal{D}_j$



n i.i.d additive bidders, m items

Revenue Maximizing Auction: Our Setting

$v_i \sim \mathcal{D} = \times_j \mathcal{D}_j$

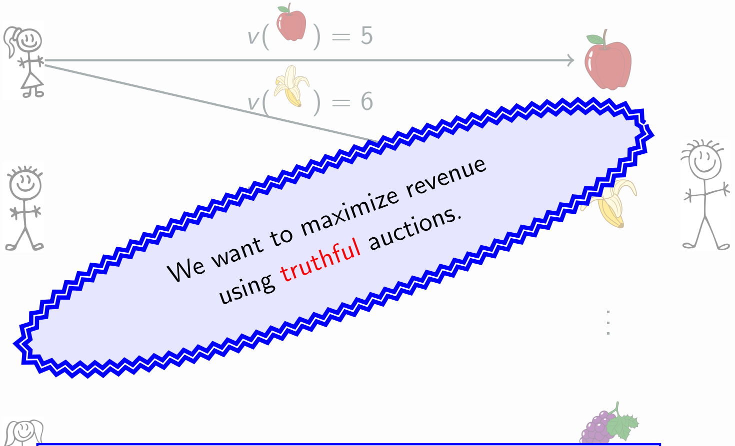


We want to maximize revenue using truthful auctions.

n i.i.d additive bidders, m items

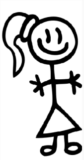
Revenue Maximizing Auction: Our Setting

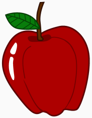
$v_i \sim \mathcal{D} = \times_j \mathcal{D}_j$



Bayesian Incentive Compatible: the bidder's expected utility is maximized by behaving truthfully when other bidders also behave truthfully

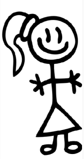
Maximizing Revenue: Single Item Setting

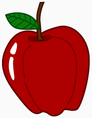




One item, one bidder:

Maximizing Revenue: Single Item Setting



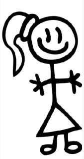


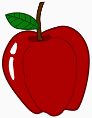
One item, one bidder:

Myerson

The optimal auction is a posted price auction.

Maximizing Revenue: Single Item Setting





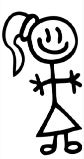
One item, one bidder:

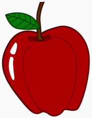
Myerson

The optimal auction is a posted price auction.

Example: $v \sim U[0, 1]$

Maximizing Revenue: Single Item Setting





One item, one bidder:

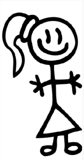
Myerson

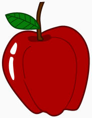
The optimal auction is a posted price auction.

Example: $v \sim U[0,1]$

Revenue from selling item at price p: $p \cdot \Pr[\text{value } \geq p] = p(1-p)$

Maximizing Revenue: Single Item Setting





One item, one bidder:

Myerson

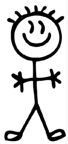
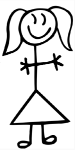
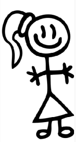
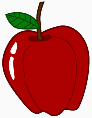
The optimal auction is a posted price auction.

Example: $v \sim U[0,1]$

Revenue from selling item at price p: $p \cdot \Pr[\text{value } \geq p] = p(1-p)$

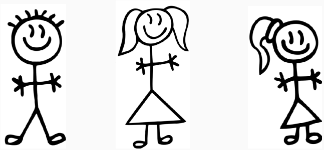
Optimal auction: sell item at price $p = 1/2$

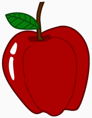
Maximizing Revenue: Single Item Setting

One item, multiple bidders:

Maximizing Revenue: Single Item Setting





One item, multiple bidders:

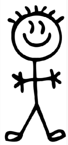
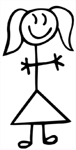
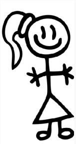
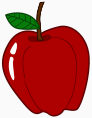
Let F be the c.d.f of D, let f be the p.d.f of D

(each bidder i's value $v_i \sim D$)

Myerson

The optimal auction maximizes the expected Myerson virtual value $\varphi(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ of the bidder that gets the item.

Maximizing Revenue: Single Item Setting

One item, multiple bidders:

Let F be the c.d.f of D, let f be the p.d.f of D

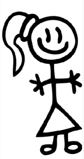
(each bidder i's value $v_i \sim D$)

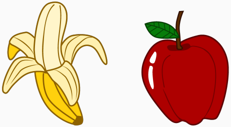
Myerson

The optimal auction maximizes the expected Myerson virtual value $\varphi(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ of the bidder that gets the item.

When the virtual value function is *regular*, the optimal auction is second price auction with reserve.

Maximizing Revenue: Multiple Item Setting



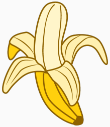


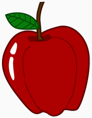
Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Maximizing Revenue: Multiple Item Setting





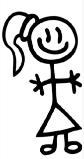
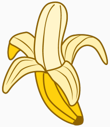
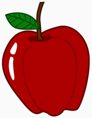
Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Optimal deterministic auction: selling separately   Revenue: 2

Maximizing Revenue: Multiple Item Setting

Example: two items, one bidder

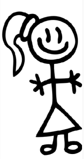
$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

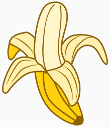
$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

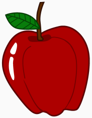
Optimal deterministic auction: selling separately Revenue: 2

Menu (p_1, p_2): get item 1 w.p. p_1, get item 2 w.p. p_2

Maximizing Revenue: Multiple Item Setting







Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Optimal deterministic auction: selling separately   Revenue: 2

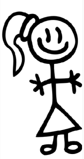
Menu (p_1, p_2): get item 1 w.p. p_1, get item 2 w.p. p_2

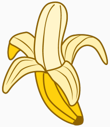
Randomize auction: $(1, \epsilon)$ with price $2 + \epsilon$

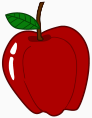
$(\epsilon, 1)$ with price $4 + \epsilon$

$(1, 1)$ with price $6 - 3\epsilon$

Maximizing Revenue: Multiple Item Setting







Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Optimal deterministic auction: selling separately Revenue: 2

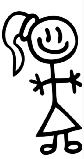
Menu (p_1, p_2): get item 1 w.p. p_1, get item 2 w.p. p_2

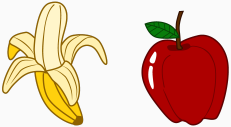
Randomize auction: $(1, \epsilon)$ with price $2 + \epsilon$ preferred by $v_1 = 2, v_2 = 1$

$(\epsilon, 1)$ with price $4 + \epsilon$

$(1, 1)$ with price $6 - 3\epsilon$

Maximizing Revenue: Multiple Item Setting





Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Optimal deterministic auction: selling separately Revenue: 2

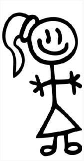
Menu (p_1, p_2): get item 1 w.p. p_1, get item 2 w.p. p_2

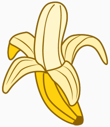
Randomize auction: $(1, \epsilon)$ with price $2 + \epsilon$ preferred by $v_1 = 2, v_2 = 1$

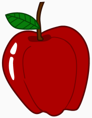
$(\epsilon, 1)$ with price $4 + \epsilon$ preferred by $v_1 = 1, v_2 = 4$

$(1, 1)$ with price $6 - 3\epsilon$

Maximizing Revenue: Multiple Item Setting







Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Optimal deterministic auction: selling separately Revenue: 2

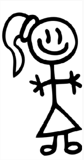
Menu (p_1, p_2): get item 1 w.p. p_1, get item 2 w.p. p_2

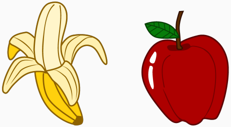
Randomize auction: $(1, \epsilon)$ with price $2 + \epsilon$ preferred by $v_1 = 2, v_2 = 1$

$(\epsilon, 1)$ with price $4 + \epsilon$ preferred by $v_1 = 1, v_2 = 4$

$(1, 1)$ with price $6 - 3\epsilon$ preferred by $v_1 = 2, v_2 = 4$

Maximizing Revenue: Multiple Item Setting





Example: two items, one bidder

$v_1 = 1$ w.p. $1/2$ and $v_1 = 2$ w.p. $1/2$

$v_2 = 1$ w.p. $3/4$ and $v_2 = 4$ w.p. $1/4$

Optimal deterministic auction: selling separately   Revenue: 2

Menu (p_1, p_2): get item 1 w.p. p_1, get item 2 w.p. p_2

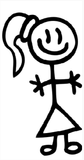
Randomize auction: $(1, \epsilon)$ with price $2+\epsilon$ preferred by $v_1 = 2, v_2 = 1$

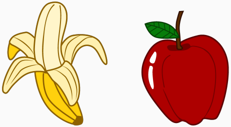
$(\epsilon, 1)$ with price $4+\epsilon$ preferred by $v_1 = 1, v_2 = 4$

$(1, 1)$ with price $6-3\epsilon$ preferred by $v_1 = 2, v_2 = 4$

Revenue: $\frac{1}{2} \cdot \frac{3}{4}(2+\epsilon) + \frac{1}{2} \cdot \frac{1}{4}(4+\epsilon) + \frac{1}{2} \cdot \frac{1}{4}(6-3\epsilon) = 2 + \frac{\epsilon}{8}$

Maximizing Revenue: Multiple Item Setting





Revenue optimal auctions are messy when $m > 1$:

- (Non-monotonicity) It might get less revenue from bidders with higher values. [HR15]
- (Randomness) It might sell "lottery tickets" for sets of items. [Tha04, MV07, Pav11, DDT17]
- (Intractability) It might present uncountably infinite number of "lottery tickets". [HN13, DDT14]

Approximating Revenue Is Possible But With Unsatisfactory Constants

Paper	n	m	Bidder Type	Approximation Ratio
[BILW14]	$n = 1$	arbitrary	additive	6
[CDW16]	arbitrary	arbitrary	additive	8
[GK16]	arbitrary	arbitrary	additive, regular	200 (prior-independent auction)
[CZ17]	arbitrary	arbitrary	XOS	268
[CZ17]	arbitrary	arbitrary	subadditive	$\log m$
		...		

Enhanced Competition

Can we get $(1-\epsilon)$ fraction of the revenue with a simple auction?

Enhanced Competition

Can we get $(1-\epsilon)$ fraction of the revenue with a simple auction?

Yes! With enhanced competition!

Enhanced Competition

Can we get $(1-\epsilon)$ fraction of the revenue with a simple auction?

Yes! With enhanced competition!

Enhanced Competition

Find number of bidders $n' > n$ where a simple auction with n' bidders (almost) match the revenue of the optimal auction with n bidders.

Enhanced Competition

Can we get $(1-\epsilon)$ fraction of the revenue with a simple auction?

Yes! With enhanced competition!

Enhanced Competition

Find number of bidders $n' > n$ where a simple auction with n' bidders (almost) match the revenue of the optimal auction with n bidders.

- **Motivation:** Instead of spending effort designing the optimal (or close to optimal) auction, spend effort recruiting bidders!

Enhanced Competition

Can we get $(1-\epsilon)$ fraction of the revenue with a simple auction?

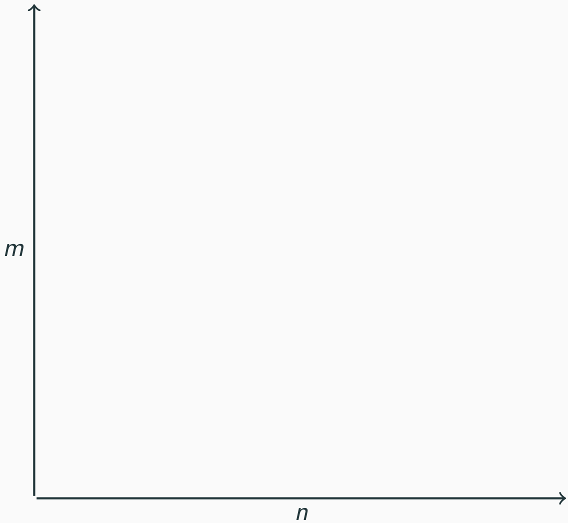
Yes! With enhanced competition!

Enhanced Competition

Find number of bidders $n' > n$ where a simple auction with n' bidders (almost) match the revenue of the optimal auction with n bidders.

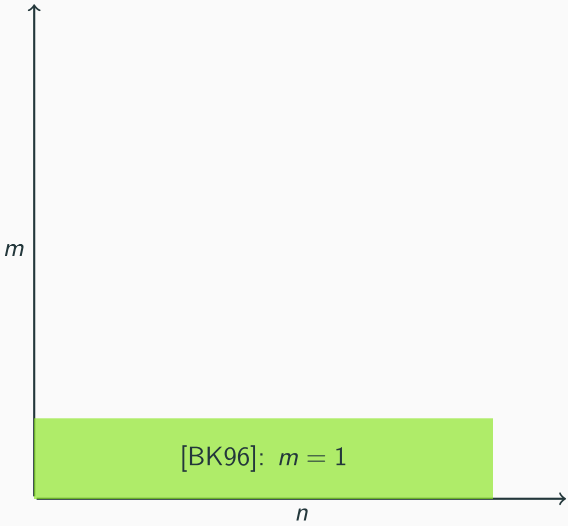
- **Motivation:** Instead of spending effort designing the optimal (or close to optimal) auction, spend effort recruiting bidders!
- **Focus of our paper: constant enhanced competition** – Is it possible to use only $n' = O(n)$ bidders?

Progress on Constant Enhanced Competition



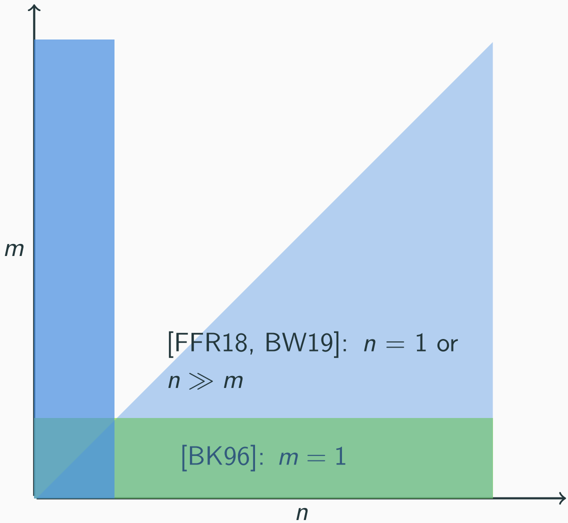
For which n, m is constant enhanced competition enough to get almost full revenue?

Progress on Constant Enhanced Competition



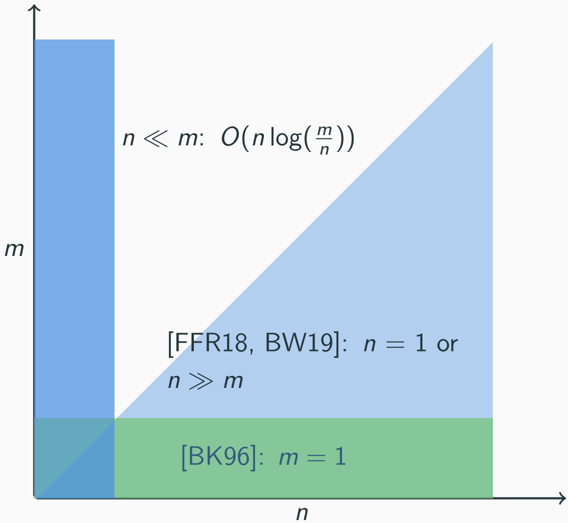
For which n, m is constant enhanced competition enough to get almost full revenue?

Progress on Constant Enhanced Competition



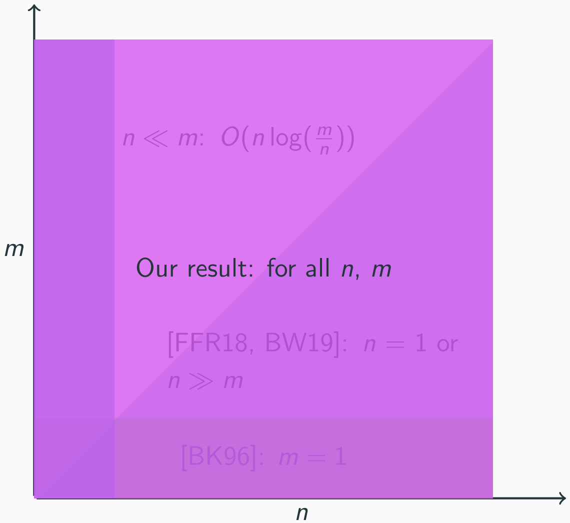
For which n, m is constant enhanced competition enough to get almost full revenue?

Progress on Constant Enhanced Competition



For which n, m is constant enhanced competition enough to get almost full revenue?

Progress on Constant Enhanced Competition



For which n, m is constant enhanced competition enough to get almost full revenue?

Our Results

Theorem 1 (informal)

A simple auction with $n' = O(n/\epsilon)$ bidders can obtain a $(1-\epsilon)$ fraction of the optimal revenue with n bidders.

Our Results

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1 - \epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction (either selling the items separately or a second price auction with an entry fee) with n' bidders generates more revenue than the optimal auction with n bidders.

Our Results

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction (either selling the items separately or a second price auction with an entry fee) with n' bidders generates c times more revenue than the optimal auction with n bidders.

Our Results

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. Any auction that guarantees a constant approximation to the optimal revenue with n' bidders generates c times more revenue than the optimal auction with n bidders.

Our Results

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. Any auction that guarantees a constant approximation to the optimal revenue with n' bidders generates c times more revenue than the optimal auction with n bidders.

Note: for any auction (or best of a group of auctions) to get near optimal revenue with constant enhance competition, it is **necessary** for the auction to guarantee a constant fraction of the optimal revenue. Our result can be viewed as saying this is **sufficient** as well.

Our Results for Regular Distributions

We will propose a **prior independent auction** that generates almost optimal revenue with constant enhanced competition.

Our Results for Regular Distributions

We will propose a **prior independent auction** that generates almost optimal revenue with constant enhanced competition.

Theorem 2

Let $\epsilon > 0$ and $n' = O(n/\epsilon^2)$. When the items are **regular**, at least one of the following hold:

1. A $(1 - \epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A prior-independent second price auction with an entry fee with n' bidders generates $\frac{1}{\epsilon}$ times more revenue than the optimal auction with n bidders.

Our Results for Regular Distributions

Theorem 2

Let $\epsilon > 0$ and $n' = O(n/\epsilon^2)$. When the items are **regular**, at least one of the following hold:

1. A $(1 - \epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A prior-independent second price auction with an entry fee and n' bidders generates $\frac{1}{\epsilon}$ times more revenue than the optimal auction with n bidders.

Hybrid Auction:

- runs second price auction w.p. $1 - \epsilon$
- runs prior-independent second price auction with an entry fee w.p. ϵ

Our Results for Regular Distributions

Theorem 2

Let $\epsilon > 0$ and $n' = O(n/\epsilon^2)$. When the items are **regular**, at least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A prior-independent second price auction with an entry fee and n' bidders generates $\frac{1}{\epsilon}$ times more revenue than the optimal auction with n bidders.

Hybrid Auction:

- runs second price auction w.p. $1-\epsilon$
- runs prior-independent second price auction with an entry fee w.p. ϵ

The hybrid auction with $n' = O(n/\epsilon^2)$ bidders obtains $(1-\epsilon)^2$-fraction of the optimal revenue with n bidders.

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1 - \epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove:

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1 - \epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove:

- **Step one:** The optimal welfare with n' bidders is much larger than the optimal welfare with n bidders.

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove:

- **Step one:** The optimal welfare with n' bidders is much larger than the optimal welfare with n bidders.
- **Step two:** The optimal virtual welfare with n' bidders is much larger than the optimal virtual welfare with n bidders.

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove:

- **Step one:** The optimal welfare with n' bidders is much larger than the optimal welfare with n bidders.
- **Step two:** The optimal virtual welfare with n' bidders is much larger than the optimal virtual welfare with n bidders.
- **Step three:** Use connection between optimal revenue and virtual welfare.

Does welfare grow with number of bidders?

Lemma

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders),

then (welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (welfare with n bidders).

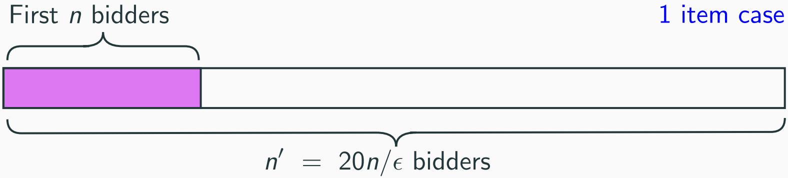
Does welfare grow with number of bidders?

Lemma

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1 - \epsilon)$ fraction of optimal revenue with n bidders),

then (welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (welfare with n bidders).

1 item case



Does welfare grow with number of bidders?

Lemma

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders),

then (welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (welfare with n bidders).

1 item case

$$
\begin{aligned}
&\mathbb{E}[\text{welfare with } n] \\
\leq &\frac{\epsilon}{20}\cdot\mathbb{E}[\text{welfare with } n'] + \left(1-\frac{\epsilon}{20}\right)\cdot\mathbb{E}[\text{2nd highest value with } n']
\end{aligned}
$$

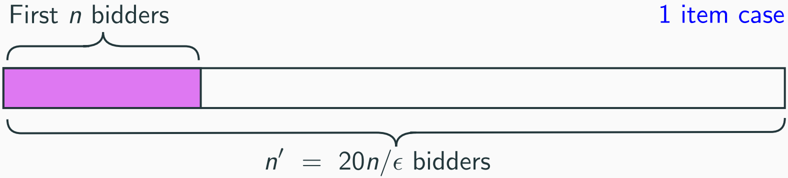
Does welfare grow with number of bidders?

Lemma

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders),

then (welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (welfare with n bidders).

1 item case



$$
\begin{aligned}
&\mathbb{E}[\text{welfare with } n] \\
\leq &\frac{\epsilon}{20} \cdot \mathbb{E}[\text{welfare with } n'] + \left(1 - \frac{\epsilon}{20}\right) \cdot \mathbb{E}[\text{2nd highest value with } n'] \\
\leq &\frac{\epsilon}{20} \cdot \mathbb{E}[\text{welfare with } n'] + (1-\epsilon)\, \mathbb{E}[\text{welfare with } n]
\end{aligned}
$$

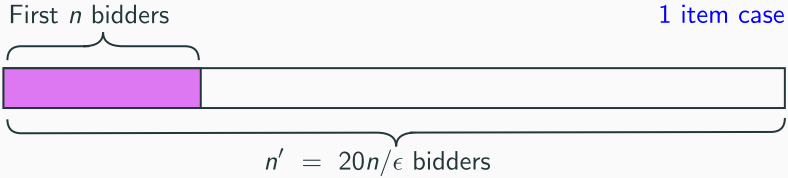
Does welfare grow with number of bidders?

Lemma

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders),

then (welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (welfare with n bidders).

1 item case



$$\begin{aligned}
&\mathbb{E}[\text{welfare with } n] \\
\leq& \frac{\epsilon}{20} \cdot \mathbb{E}[\text{welfare with } n'] + \left(1 - \frac{\epsilon}{20}\right) \cdot \mathbb{E}[\text{2nd highest value with } n'] \\
\leq& \frac{\epsilon}{20} \cdot \mathbb{E}[\text{welfare with } n'] + (1-\epsilon)\,\mathbb{E}[\text{welfare with } n] \\
\Rightarrow& \epsilon \cdot \mathbb{E}[\text{welfare with } n] \leq \frac{\epsilon}{20} \cdot \mathbb{E}[\text{welfare with } n']
\end{aligned}$$

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove

- **Step one:** The optimal welfare with n' bidders is much larger than the optimal welfare with n bidders.
- **Step two:** The optimal virtual welfare with n' bidders is much larger than the optimal virtual welfare with n bidders.
- **Step three:** Use connection between optimal revenue and virtual welfare.

Connection Between Optimal Revenue And Virtual Welfare

[CDW16]

For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee).

Connection Between Optimal Revenue And Virtual Welfare

[CDW16]

For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee).

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1 - \epsilon)$ fraction of optimal revenue with n bidders),

then (virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20 \cdot$ (virtual welfare with n bidders).

Connection Between Optimal Revenue And Virtual Welfare

[CDW16]

For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee).

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders),

then (virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Proof of Theorem 1

revenue from n bidders $\leq$ virtual welfare from n bidders

Connection Between Optimal Revenue And Virtual Welfare

[CDW16]

For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee).

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1 - \epsilon)$ fraction of optimal revenue with n bidders),

then (virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Proof of Theorem 1

$$\begin{aligned} \text{revenue from } n \text{ bidders} &\leq \text{virtual welfare from } n \text{ bidders} \\ &\leq \frac{1}{20} \cdot \text{virtual welfare from } n' \text{ bidders} \end{aligned}$$

Connection Between Optimal Revenue And Virtual Welfare

[CDW16]

For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee).

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders),

then (virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Proof of Theorem 1

$$\begin{aligned}
\text{revenue from } n \text{ bidders } &\leq \text{virtual welfare from } n \text{ bidders} \\
&\leq \frac{1}{20} \cdot \text{virtual welfare from } n' \text{ bidders} \\
&\leq \frac{8}{20} \cdot \text{revenue from a simple auction with } n' \text{ bidders}
\end{aligned}$$

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove:

- **Step one:** The optimal welfare with n' bidders is much larger than the optimal welfare with n bidders.
- **Step two:** The optimal virtual welfare with n' bidders is much larger than the optimal virtual welfare with n bidders.
- **Step three:** Use connection between optimal revenue and virtual welfare.

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1 - \epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Virtual value from [CDW16]:

$$\Phi_j^n(v_i, v_{-i}) = \begin{cases} \tilde{\varphi}_j(v_{i,j})^+ & \text{if bidder } i \text{ gains the highest (and non-negative)} \\ & \text{utility from item } j \text{ in second price auction} \\ v_{i,j} & \text{otherwise} \end{cases}$$

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Virtual value from [CDW16]:

$$\Phi_j^n(v_i, v_{-i}) = v_{i,j} \cdot \mathbb{1}\left(v_i \notin R_j^{v_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in R_j^{v_{-i}}\right).$$

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Virtual value from [CDW16]:

$$\Phi_j^n(v_i, v_{-i}) = v_{i,j} \cdot \mathbb{1}\left(v_i \notin R_j^{v_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in R_j^{v_{-i}}\right).$$

Conditions:

- The virtual values must be at most the corresponding values.
- The distribution of virtual values for different bidders are independent and identical.
- The distribution of virtual values does not depend on the number n of bidders participating in the auction.

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Virtual value from [CDW16]:

$$\Phi_j^n(v_i, v_{-i}) = v_{i,j} \cdot \mathbb{1}\left(v_i \notin R_j^{v_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in R_j^{v_{-i}}\right).$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical.
- The distribution of virtual values does not depend on the number n of bidders participating in the auction.

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Virtual value from [CDW16]:

$$\Phi_j^n(v_i, v_{-i}) = v_{i,j} \cdot \mathbb{1}\left(v_i \notin R_j^{v_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in R_j^{v_{-i}}\right).$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical. ✗
- The distribution of virtual values does not depend on the number n of bidders participating in the auction.

Does virtual value grow with number of bidders?

Lemma Modified

Assume event A (second price auction with $n' = \frac{20n}{\epsilon}$ bidders extract at most $(1-\epsilon)$ fraction of optimal revenue with n bidders), then

(virtual welfare with $n' = \frac{20n}{\epsilon}$ bidders) $\geq 20\cdot$ (virtual welfare with n bidders).

Virtual value from [CDW16]:

$$\Phi_j^n(v_i, v_{-i}) = v_{i,j} \cdot \mathbb{1}\left(v_i \notin R_j^{v_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathbb{1}\left(v_i \in R_j^{v_{-i}}\right).$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical. ✗
- The distribution of virtual values does not depend on the number n of bidders participating in the auction. ✗

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

A modified virtual value:

$$\Phi_j^n(v_i) = \mathop{\mathbb{E}}_{v_{-i}\sim\mathcal{D}^{n-1}}\left[\Phi_j^n(v_i, v_{-i})\right]$$

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \notin R_j^{v_{-i}} \right) \right] + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \in R_j^{v_{-i}} \right) \right].$$

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \notin R_j^{v_{-i}} \right) \right] + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \in R_j^{v_{-i}} \right) \right].$$

Conditions:

- The virtual values must be at most the corresponding values.
- The distribution of virtual values for different bidders are independent and identical.
- The distribution of virtual values does not depend on the number n of bidders participating in the auction.

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \notin R_j^{v_{-i}} \right) \right] + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \in R_j^{v_{-i}} \right) \right].$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical.
- The distribution of virtual values does not depend on the number n of bidders participating in the auction.

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \notin R_j^{v_{-i}} \right) \right] + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \in R_j^{v_{-i}} \right) \right].$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical. ✓
- The distribution of virtual values does not depend on the number n of bidders participating in the auction.

Redefining Virtual Value

Idea: take the expectation (draw $n-1$ ghost bidders)

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \notin R_j^{v_{-i}} \right) \right] + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{v_{-i} \sim \mathcal{D}^{n-1}} \left[\mathbb{1}\left(v_i \in R_j^{v_{-i}} \right) \right].$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical. ✓
- The distribution of virtual values does not depend on the number n of bidders participating in the auction. ✗

Redefining Virtual Value

Idea: fix some large $n' > n$, draw $n' - 1$ bidder values $w_{-i} \sim \mathcal{D}^{n'-1}$

Redefining Virtual Value

Idea: fix some large $n' > n$, draw $n' - 1$ bidder values $w_{-i} \sim \mathcal{D}^{n'-1}$

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \notin R_j^{w_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \in R_j^{w_{-i}}\right).$$

Redefining Virtual Value

Idea: fix some large $n' > n$, draw $n' - 1$ bidder values $w_{-i} \sim \mathcal{D}^{n'-1}$

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \notin R_j^{w_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \in R_j^{w_{-i}}\right).$$

Redefining Virtual Value

Idea: fix some large $n' > n$, draw $n' - 1$ bidder values $w_{-i} \sim \mathcal{D}^{n'-1}$

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \notin R_j^{w_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \in R_j^{w_{-i}}\right).$$

Conditions:

- The virtual values must be at most the corresponding values. ✓
- The distribution of virtual values for different bidders are independent and identical. ✓
- The distribution of virtual values does not depend on the number n of bidders participating in the auction. ✓

Redefining Virtual Value

Idea: fix some large $n' > n$, draw $n' - 1$ bidder values $w_{-i} \sim \mathcal{D}^{n'-1}$

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \notin R_j^{w_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \in R_j^{w_{-i}}\right).$$

[CDW16]

For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee).

Redefining Virtual Value

Idea: fix some large $n' > n$, draw $n' - 1$ bidder values $w_{-i} \sim \mathcal{D}^{n'-1}$

A modified virtual value:

$$\Phi_j^n(v_i) = v_{i,j} \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \notin R_j^{w_{-i}}\right) + \tilde{\varphi}_j(v_{i,j})^+ \cdot \mathop{\mathbb{E}}_{w_{-i} \sim \mathcal{D}^{n'-1}} \mathbb{1}\left(w_i \in R_j^{w_{-i}}\right).$$

[CDW16]
For any fixed number of bidders N, the optimal revenue is at most the expected virtual welfare, which is at most 8 times revenue from a simple auction (selling separately or second price with entry fee). ✓

Theorem 1 Proof Outline

Theorem 1

Let $\epsilon > 0$ and $n' = O(n/\epsilon)$. At least one of the following hold:

1. A $(1-\epsilon)$-fraction of the optimal revenue with n bidders is obtained by a second price auction with n' bidders.
2. A simple auction with n' bidders generates more revenue than the optimal revenue with n bidders.

Event A: case (1) does not hold

Assuming event A, we prove

- **Step one:** The optimal welfare with n' bidders is much larger than the optimal welfare with n bidders.
- **Step two:** The optimal virtual welfare with n' bidders is much larger than the optimal virtual welfare with n bidders.
- **Step three:** Use connection between optimal revenue and virtual welfare.

Conclusion

We show that, for all m and n, an arbitrarily large constant fraction of the optimal revenue from selling m items to n bidders can be obtained via simple auctions with $O(n)$ bidders.

Conclusion

We show that, for all m and n, an arbitrarily large constant fraction of the optimal revenue from selling m items to n bidders can be obtained via simple auctions with $O(n)$ bidders.

Future directions:

- Obtains full optimal revenue with $O(n)$ bidders?
- Obtain almost optimal revenue with $n + o(n)$ bidders or prove a lower bound?
- Our work can also be viewed as proving for additive valuations an equivalence between auctions that gets a constant fraction of the optimal revenue and auctions that has $O(n)$ enhanced competition. Can we prove this for more general class of valuation functions?

Thank you!

Questions?

References i

- Moshe Babaioff, Nicole Immorlica, Brendan Lucier, and S. Matthew Weinberg, *A simple and approximately optimal mechanism for an additive buyer*, 55th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2014, Philadelphia, PA, USA, October 18-21, 2014, 2014, pp. 21–30.
- Jeremy Bulow and Paul Klemperer, *Auctions versus negotiations*, The American Economic Review (1996), 180–194.
- Hedyeh Beyhaghi and S. Matthew Weinberg, *Optimal (and benchmark-optimal) competition complexity for additive buyers over independent items*, Proceedings of the 51st ACM Symposium on Theory of Computing Conference (STOC), 2019.

Yang Cai, Nikhil Devanur, and S. Matthew Weinberg, *A duality based unified approach to bayesian mechanism design*, Proceedings of the 48th ACM Conference on Theory of Computation(STOC), 2016.

Yang Cai and Mingfei Zhao, *Simple mechanisms for subadditive buyers via duality*, Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017, 2017, pp. 170–183.

Constantinos Daskalakis, Alan Deckelbaum, and Christos Tzamos, *The Complexity of Optimal Mechanism Design*, the 25th ACM-SIAM Symposium on Discrete Algorithms (SODA), 2014.

______, *Strong duality for a multiple-good monopolist*, Econometrica **85** (2017), no. 3, 735–767.

- Michal Feldman, Ophir Friedler, and Aviad Rubinstein, *99% revenue via enhanced competition*, Proceedings of the 2018 ACM Conference on Economics and Computation, Ithaca, NY, USA, June 18-22, 2018, 2018, pp. 443–460.
- Kira Goldner and Anna R Karlin, *A prior-independent revenue-maximizing auction for multiple additive bidders*, Proceedings of the 12th International Conference on Web and Internet Economics-Volume 10123, 2016, pp. 160–173.
- Sergiu Hart and Noam Nisan, *The menu-size complexity of auctions*, the 14th ACM Conference on Electronic Commerce (EC), 2013.
- Sergiu Hart and Philip J. Reny, *Maximizing Revenue with Multiple Goods: Nonmonotonicity and Other Observations*, Theoretical Economics **10** (2015), no. 3, 893–922.

References iv

A. M. Manelli and D. R. Vincent, *Multidimensional Mechanism Design: Revenue Maximization and the Multiple-Good Monopoly*, Journal of Economic Theory **137** (2007), no. 1, 153–185.

Gregory Pavlov, *Optimal mechanism for selling two goods*, The B.E. Journal of Theoretical Economics **11** (2011), no. 3.

John Thanassoulis, *Haggling over substitutes*, Journal of Economic Theory **117** (2004), 217–245.